Extremal and Probabilistic Graph Theory
Lecture 9
March 29, Tuesday

Recall that $\mathcal{F}$ is a family of graphs, we have:

Theorem 7.1 (Erdős-Stone-Simonovits)**.**

$$ex(n,\mathcal{F}) = (1 - \frac{1}{\chi(\mathcal{F})-1} + o(1))\binom{n}{2}$$

Definition 7.2. The *edit-distance* $d(G,H)$ of 2 graphs G and H with the same vertex set is the minimum k such that G can be obtained from H by adding or deleting k edges.

Remark. $d(G,H) = \min_\pi |E(G_\pi)\Delta E(H_\pi)|$ where $\pi : V \to V$ is a bijection.

Theorem 7.3 (Erdős-Simonovits Stability Theorem)**.** *For $\forall \varepsilon > 0$, and a family $\mathcal{F}$ of graphs with $\chi(\mathcal{F}) = r+1$, there is a $\delta > 0$ and n_0 such that if G is $\mathcal{F}-free$ with at least n_0 vertices, then $e(G) \geq (1-\frac{1}{r})\binom{n}{2} - \delta n^2$ implies that G is "close" to $T_r(n)$.*

Remark. This theorem means $d(G, T_r(n)) \leq \varepsilon n^2$.

We prove the following version which is for $\mathcal{F} = \{K_{r+1}\}$

Theorem 7.4 (Füredi,2015)**.** *If G is an n-vertex $K_{r+1}-free$ graph with $e(G) = e(T_r(n)) - t$, then there exists an r-partite subgraph H of G such that $e(H) \geq e(G) - t$.*

Remark. Note that t here is arbitrary! We don't need any assumption on t.

Proof. We will use the so-called Erdös degree majorization algorithm, which will find a partition $V(G) = V_1 \cup V_2 \cup \cdots V_r$ such $\sum_{i=1}^r e(G[V_i]) \leq t$ ($S \subset G, G[S]$ is the induced subgraph on S of G).

Let $x_1 \in V(G)$ be a vertex of max degree. Let $V_1^+ = N(x_1)$ and $V_1 = V(G)/V_1^+$. Then

$$\sum_{v\in V_1} d_G(v) \leq \sum_{u\in V_1} d(x_1) = |v_1||v_1^+|$$

$$\sum_{v\in V_1} d_G(v) = 2e(G[V_1]) + e(V_1, V_1^+)$$

In general, let x_i be a vertex in $G[V_{i-1}^+]$ of max degree. Let $V_i^+ = V_{i-1}^+ \cap N_G(x_i)$ and $V_i = V_{i-1}^+/V_i^+$, then

$$2e(G[V_i]) + e(V_i, V_i^+) = \sum_{v\in V_i} d_{G[V_{i-1}^+]}(v) \leq |V_i^+||V_i|$$

This procedure must terminate, say in s steps (when V_{s-1}^+ is an independent set, then $V_s = V_{s-1}^+$). Thus we get $V(G) = V_1 \cup V_2 \cup \cdots \cup V_s$ and s vertices $x_1, x_2, \cdots, x_s$ where $x_i \in V_i$. By the algorithm, $G[\{x_1, x_2, \cdots, x_s\}] = K_s$, so $s \leq r$. Note that $V_i^+ = V - V_1 \cup V_2 \cup \cdots \cup V_i$, so

$$2e(G[V_i]) + e(V_i, \cup_{j>i} V_j) \leq |V_i||\cup_{j>i} V_j|$$

Summing over $1 \leq i \leq s$, we have

$$e(G) + \sum_{i=1}^{s} e(G[V_i]) \leq \sum_{i=1}^{s} |V_i|| \cup_{j>i} V_j| = e(K_{V_1,V_2,\cdots,V_s}) \leq e(T_r(n))$$

So

$$\sum_{i=1}^{s} e(G[V_i]) \leq e(T_r(n)) - e(G) = t.$$

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Remark. Here, some V_i is allowed to be empty

Corollary 7.5 (Stability for K_{r+1}). *Suppose G is $K_{r+1} - free$ with $e(G) \geq e(T_r(n)) - t$, then there is a complete r-graph $K = K_{V_1,V_2,\cdots,V_r}$ with $V(G) = \cup_{i=1}^{r} V_i$ satisfying $d(G,K) \leq 3t$.*

Proof. Left as an exercise. Note that

$$e(H) \geq e(G) - t = e(T_r(n)) - 2t \geq e(K_{V_1,V_2,\cdots,V_r}) - 2t$$

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The stability method, in rough terms, says the following: we first determine the appropriate structure of near extremal graphs (usually achieved by stability theorems), and then use this approximate structure to obtain exact structure of exactly extremal graphs.

Definition 7.6. A graph F is r-edge-critical if there exists an edge e such that $\chi(F - e) < \chi(F) = r$.

Example 7.7. $K_r, C_{2k+1}, \cdots$

Recall that Turán Theorem asserts that $ex(n, K_{r+1}) = e(T_r(n))$ and the unique extremal graph is $T_r(n)$.

Next, we will use the following theorem as a running example of the stability method, which shows that the extremal graph for edge-critical graphs F is also the Turán graph.

Theorem 7.8. *Let F be an $r+1$-edge-critical graph for $r \geq 2$. Then $ex(n, F) = e(T_r(n))$ and the unique extremal graph is $T_r(n)$ provided that n is sufficiently large (say $n \geq n_0(F)$).*